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An Iteration Perturbation Technique

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A numerical iteration technique, which is simple to use and easy to implement, is proposed to solve perturbation problems. The technique may be used to supplement or to complement the PLK (Poincare, Lighthill, and Kuo) method. The iteration perturbation technique can solve problems for which the PLK method fails. Numerical experience with several examples is described.

1. INTRODUCTION

Differential equations of the form

$$(f(x) + \epsilon y)(dy/dx) + q(x)\, y = r(x), \qquad 0 \leqslant x \leqslant 1, \tag{1.1}$$

$$y(1) = c, \tag{1.2}$$

where c is a prescribed constant and ϵ is a small (perturbation) parameter are of common occurrence in fluid mechanics and other branches of applied mathematics. For example, the problem of the shock waves produced by a circular cylinder expanding uniformly in still air leads to a second order nonlinear differential equation for the velocity potential [7]. When this equation is written as two first-order differential equations, one of the equations is trivial and the other can be brought into the form of (1.1).

In the most widely studied cases of (1.1), where $f(x) = x^n$, n an integer $\geqslant 1$, problems of the form (1.1)–(1.2) can not be solved by the usual method of expanding the solution in a perturbation series

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots, \tag{1.3}$$

substituting (1.3) into (1.1), equating like powers of ϵ, and solving the resulting sequence of problems for $y_0(x)$, $y_1(x)$,... . For as Comstock [5] points out, while the original equation has no singularity at the origin, not only does the equation for $y_0(x)$ have a singularity at the origin, but the solutions for $y_1(x)$, $y_2(x)$,..., are successively more singular.

Lighthill's method [8] (commonly referred to as the PLK method for Poincare, Lighthill, and Kuo) of overcoming this difficulty was to expand the independent as well as the dependent variable in a perturbation series

$$x = z + \epsilon x_1(z) + \epsilon^2 x_2(z) + \cdots, \tag{1.4}$$

$$y = y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \cdots, \tag{1.5}$$

which effectively moves the singularities in y_0, y_1,..., outside of the interval of interest $0 \leqslant x \leqslant 1$. One of the reasons that the PLK method is so important is that through it an analytical expression can be obtained for $y(x)$ in the vicinity of the origin. A discussion of the PLK method and subsequent developments and generalizations can be found in Comstock's survey article [5] and in [2].

Now, as Comstock [4, 5] points out, there are cases in which the PLK method appears to work but in fact gives quantitatively erroneous results. For such problems, and for problems in which it may be too difficult to obtain analytic (closed form) solutions, we propose in this paper an iterative numerical solution scheme. While our numerical scheme does not have the desirable feature of furnishing an analytical expression for the behavior of $y(x)$ at the origin, it has given correct numerical solutions when the PLK method has failed. Such solutions for several values of ϵ combined with an extrapolation technique can yield the behavior of $y(x)$ at the origin, as we show by an example.

The method is described in detail in the following section, and numerical experience with three perturbation problems is presented in Section 3.

2. Iteration Technique

The solution technique is a numerical one. It consists of generating a sequence of solutions $y_0(x)$, $y_1(x)$,..., to the original problem where $y_0(x)$ is obtained by setting $\epsilon = 0$. The $y_0(x)$ solution sometimes may be found analytically, or by numerical integration. The solution for $y_1(x)$ is generated by numerical integration of the differential equation where $y_0(x)$ appears on the right-hand side of the differential equation and ϵ takes its prescribed numerical value. The solution for $y_2(x)$ is found similarly in terms of $y_1(x)$ and the prescribed value of ϵ, etc. A variety of possibilities exist for the iteration form of the differential equations. This is best exhibited by an example.

Consider the perturbation equation

$$(f(x) + \epsilon y)(dy/dx) + q(x)\, y = r(x). \tag{2.1}$$

For the $y_0(x)$ equation, set $\epsilon = 0$, and solve

$$f(x)(dy_0/dx) + q(x)\, y_0 = r(x) \tag{2.2}$$

for $y_0(x)$.

For $y_n(x)$, $n = 1, 2,...$, the equation may be put into three forms. The Method I form is

$$dy_n/dx = (r(x) - q(x)\, y_{n-1})/(f(x) + \epsilon y_{n-1}) \tag{2.3}$$

where only y_{n-1} appears on the right-hand side in both the numerator and the denominator.

The Method II form is

$$dy_n/dx = (r(x) - q(x)\, y_{n-1})/(f(x) + \epsilon y_n) \tag{2.4}$$

where $y_{n-1}(x)$ appears in the numerator and $y_n(x)$ in the denominator on the right-hand side.

The Method III form is

$$dy_n/dx = (r(x) - q(x)\, y_n)/(f(x) + \epsilon y_{n-1}) \tag{2.5}$$

where $y_n(x)$ appears in the numerator and y_{n-1} in the denominator. For more complicated equations, it is possible to express the differential equation for the nth iteration in terms of both y_n and y_{n-1} in both the numerator and the denominator. In addition one may employ on the right-hand side y_n, y_{n-1},..., y_0 or some weighted combination of them.

It turns our that the simplest expression, Eq. (2.3) is really Picard's method expressed in differential equation form rather than the familiar integral equation form [6]. We can accordingly establish the conditions under which Method I converges. In general convergence will depend on the Lipschitz constant K of $[r(x) - q(x)\, y(x)]/[f(x) + \epsilon y(x)]$; namely,

$$K = \max_{0 \leqslant x \leqslant 1} \max_{Y,y} |[f(x)\, q(x) + \epsilon r(x)]/[(f(x) + \epsilon Y)(f(x) + \epsilon y)]|\,, \tag{2.6}$$

where Y, y lie in the interval containing the solution $y(x)$ of (1.1).

It should be noted that for problems of interest, the solution for $y_0(x)$ is singular at the origin. In the original computer program embodying our method, this singularity was handled by a programming device of approximating $x = 0$ by a small finite number, namely 10^{-3}. As an alternative approach, the Referee, commenting on an earlier version of this paper, suggests that since $y_0(x) \to \infty$

as $x \to 0$, then $dy_1/dx \to -q(0)/\epsilon$ as $x \to 0$, and accordingly an improved initial solution $\tilde{y}_0(x)$ may be obtained in the matched form

$$\tilde{y}_0(x) = \begin{cases} -q(0)\, x/\epsilon + \bar{k}, & 0 \leqslant x \leqslant \bar{x}, \\ y_0(x), & \bar{x} \leqslant x \leqslant 1. \end{cases} \tag{2.7}$$

The parameters $\bar{x}$ and $\bar{k}$ are determined from the equations expressing the continuity of the solution and the slope

$$-q(0)\, \bar{x}/\epsilon + \bar{k} = y_0(\bar{x}), \tag{2.8}$$

$$-q(0)/\epsilon = dy_0(\bar{x})/dx. \tag{2.9}$$

This procedure can be employed in our Examples 1 and 3, but not in Example 2, where $q(0) = 0$ but $dy_0(\bar{x})/dx \neq 0$, $0 \leqslant x \leqslant 1$. When we tried this alternative method for obtaining $\tilde{y}_0(x)$ in Example 1, we discovered that the solution converged at a rate slightly slower than the original method. However, this difference in the rate of convergence is so small that it may not be statistically significant. We continue to believe that the alternative method has merit, since it handles the singularity condition by an analytical approach rather than a programming device, and that it could very well exhibit superior performance in other problems and/or other data.

For the PLK method Wasow [9] obtained a sufficient condition; namely,

$$y_0(z)\, q(z) - r(z) \neq 0, \qquad 0 \leqslant z \leqslant 1. \tag{2.10}$$

Since the expression $y_0(z)\, q(z) - r(z)$ appears in the denominator of Wasow's equations, the criterion (2.10) is the condition that rules out the presence of a singularity in the interval $0 \leqslant z \leqslant 1$.

In our development, the numerators in the Methods I, II, and III are of the form $r(x) - q(x)\, y_{n-1}$ or $r(x) - q(x)\, y_n$, which are similar to the left-hand side of (2.10). The Wasow condition, however, does not apply to our methods since the expressions like $r(x) - q(x)\, y_{n-1}$, being in the numerator, no longer cause singularities when they vanish. From a practical point of view this may mean that the iteration perturbation method may be a candidate for problems which can not be solved by the PLK method due to the Wasow condition not being satisfied.

3. Numerical Experience

We will discuss three examples of perturbation problems which have appeared in the literature. All the examples possess analytical solutions against which numerical solutions can be compared. They have been solved over a variety of

conditions by Method I, which we will refer to as the Picard perturbation method.

EXAMPLE 1 [5]. Consider the following example

$$(x + \epsilon u)(du/dx) = -u + 2bx, \qquad 0 \leqslant x \leqslant 1, \tag{3.1}$$

$$u(1) = B = b(1 + \eta), \tag{3.2}$$

which possesses the analytical solution

$$u = \epsilon^{-1}\{-x + (x^2 + 2\epsilon(\bar{c} + bx^2))^{1/2}\} \tag{3.3}$$

where

$$\bar{c} = b\eta + \epsilon(b^2/2)(1 + \eta)^2. \tag{3.4}$$

For $\epsilon = 0$, we obtain the analytic solution for $u_0(x)$

$$u_0(x) = bx + b\eta/x. \tag{3.5}$$

Putting the equation in the form of Method I, we have

$$du_1/dx = (-u_0 + 2bx)/(x + \epsilon u_0). \tag{3.6}$$

It is interesting to note that (3.6) also has an analytic solution

$$u_1(x) = \frac{bx}{C} - \left\{\arctan x \left(\frac{C}{D}\right)^{1/2}\right\}\left\{\frac{bD}{C(CD)^{1/2}} + \frac{b\eta}{(CD)^{1/2}}\right\} + C_1 \tag{3.7}$$

where

$$\begin{aligned} C &= 1 + \epsilon b, \\ D &= \epsilon b\eta, \\ C_1 &= B - \frac{b}{C} + \left\{\arctan \left(\frac{C}{D}\right)^{1/2}\right\}\left\{\frac{bD}{C(CD)^{1/2}} + \frac{b\eta}{(CD)^{1/2}}\right\}, \end{aligned} \tag{3.8}$$

but it is increasingly difficult to obtain analytical solutions for $u_n(x)$, $n = 2, 3, \ldots$.

The problem was solved numerically by the Picard perturbation method (Method I) and by Method II. The Picard perturbation method gave more accurate results. For the boundary condition $y(1) = B$, with $b = 1$ and $\eta = .5$ the analytical solution and the numerical integration solution for $u_5(x)$ by the Picard perturbation method are exhibited in Table 1 for $\epsilon = 1.0$, 10^{-1}, 10^{-2}, 10^{-4}, and 10^{-8}. The numerical integration was executed by a four point double precision Runge-Kutta method. From these tables we may conclude the following.

TABLE 1

Example 1. $u(1) = B = 1.5$, $b = 1.0$, $\eta = .5$

	$\epsilon = 1.0$		$\epsilon = .10$	
x	$u(x)$	$u_5(x)$	$u(x)$	$u_5(x)$
0.00	1.8027753	1.8027779	3.5000000	3.4999954
0.05	1.7548542	1.7548568	3.0425968	3.0425931
0.10	1.7110757	1.7110793	2.6674242	2.6674199
0.20	1.6357545	1.6357582	2.1291637	2.1291687
0.30	1.5761652	1.5761684	1.8010416	1.8010551
0.40	1.5313196	1.5313227	1.6080293	1.6080451
0.50	1.4999990	1.5000016	1.5000000	1.5000119
0.60	1.4808635	1.4808665	1.4464750	1.4464834
0.70	1.4725551	1.4725570	1.4291153	1.4291207
0.80	1.4737625	1.4737639	1.4366302	1.4366330
0.90	1.4832744	1.4832753	1.4618339	1.4618363
1.00	1.4999990	1.5000000	1.4999952	1.5000000
	$\epsilon = 10^{-2}$		$\epsilon = 10^{-4}$	
0.00	$1.0111874(10^1)$	$1.0117332(10^1)$	$1.0001124(10^2)$	$1.0433836(10^2)$
0.05	6.3026545	6.3062624	9.9531841	9.9846446
0.10	4.2916059	4.2931248	5.0881802	5.0922413
0.20	2.5887140	2.5890216	2.6987417	2.6992376
0.30	1.9413525	1.9414558	1.9663972	1.9665362
0.40	1.6443273	1.6443728	1.6499409	1.6499948
0.50	1.5000000	1.5000223	1.5000000	1.5000245
0.60	1.4349249	1.4349363	1.4333496	1.4333616
0.70	1.4160346	1.4160404	1.4143035	1.4143095
0.80	1.4263709	1.4263498	1.4250137	1.4250165
0.90	1.4562737	1.4562747	1.4555628	1.4555639
1.00	1.5000000	1.5000000	1.5000000	1.5000000
	$\epsilon = 10^{-8}$			
0.00	$1.0000000(10^4)$	$3.2938145(10^5)$		
0.05	$1.0049990(10^1)$	$1.0083221(10^1)$		
0.10	5.0999988	5.1041126		
0.20	2.6999998	2.7004975		
0.30	1.9666666	1.9668059		
0.40	1.6499999	1.6500540		
0.50	1.5000000	1.5000245		
0.60	1.4333333	1.4333454		
0.70	1.4142857	1.4142917		
0.80	1.4250000	1.4250028		
0.90	1.4555555	1.4555566		
1.00	1.4999999	1.5000000		

(1) The maximum relative error, $\max_{0\leqslant x\leqslant 1}(u(x) - u_n(x))/u(x)$ is of the order 10^{-4}, where $u(x)$ is the analytical solution, $n = 5$, $0 \leqslant x \leqslant 1.0$, $10^{-2} \leqslant \epsilon \leqslant 1.0$.

(2) For $\epsilon = 10^{-4}$ and 10^{-8}, the maximum relative error over the interval $0 < x \leqslant 1.0$ is of the order 10^{-3}. At $x = 0$ and $\epsilon = 10^{-4}$, the relative error is of the order .04, while for $x = 0$ and $\epsilon = 10^{-8}$, the relative error is of the order 33.

It appears, therefore, that over a wide range of ϵ the Picard perturbation method gives good agreement with the analytical solution over the entire profile including the origin, even though $u_0(x)$ is a poor approximation to the true solution. Except within the vicinity of $x = 0$, the profiles become stable after a few iterations. This suggests that for the very smallest ϵ that improved accuracy may be obtained by iterating only over the interval (.10, 0) and using a smaller step size. Furthermore, the possibility of employing extrapolation procedures appears reasonable. Two extrapolation possibilities come to mind. The first is to extrapolate $u_n(x)$ versus x to $x = 0$ for a specific ϵ and n. The second extrapolation procedure is to extrapolate $u_n(0)$ versus ϵ for a specific n. We, in fact, applied the second technique, using $u_5(0)$ for $\epsilon = 1$, 10^{-1}, 10^{-2}, 10^{-4} and extrapolating on a log–log plot to obtain a value of $u_5(0)$ at $\epsilon = 10^{-8}$ of 10^4, which is the true value.

EXAMPLE 2 [4, 5]. Consider the perturbation problem

$$(x^n + \epsilon u)(du/dx) + nx^{n-1}u - mx^{m-1} = 0, \tag{3.9}$$

$$u(1) = b \geqslant 1, \tag{3.10}$$

for which the analytical solution is

$$x^n u + (1/2)\,\epsilon u^2 = x^m + (b + (1/2)\,\epsilon b^2 - 1) \tag{3.11}$$

or

$$u(x) = \frac{-x^n \pm (x^{2n} + 2\epsilon[x^m + (b + .5\epsilon b^2 - 1)])^{1/2}}{\epsilon}. \tag{3.12}$$

To satisfy the boundary condition at $x = 1$, the sign of the discriminant must be positive.

For $\epsilon = 0$, the solution of (3.9), (3.10) is

$$u_0(x) = (x^m + b - 1)\,x^{-n}. \tag{3.13}$$

This example was solved for a variety of combinations of m and n for $b = 1$, $\epsilon = .01$; $b = 2$, $\epsilon = .01$; and for $b = 2$, $\epsilon = .0001$. In Tables 2, 3, and 4 are listed the analytical results at $x = 0$, $u(0)$, and the Picard perturbation results,

TABLE 2

Example 2. $b = 1.0$, $\epsilon = .01$

m	n	$u(0)$ analytical	$u_n(0)$	No. iterations for convergence
0.0	1.0	14.17744688	14.18032916	10
0.5	3.0	1.0	nonconvergent	
1.0	0.0	0.004999875	0.004999874	5
1.0	0.5	1.0	nonconvergent	
1.0	3.0	1.0	0.963893963	20
1.0	10.0	1.0	1.060516712	20
2.0	3.0	1.0	1.000975702	20
2.0	10.0	1.0	1.068510671	15
2.0	100.0	1.0	nonconvergent	
3.0	1.0	1.0	0.963893963	20
3.0	2.0	1.0	1.001639530	15
3.0	3.0	1.0	1.000000000	1
5.0	0.5	1.0	nonconvergent	
5.0	5.0	1.0	1.000010229	20
5.0	10.0	1.0	1.020694106	20
10.0	2.0	1.0	1.004152528	10
10.0	5.0	1.0	1.011671339	19
10.0	10.0	1.0	1.000011936	20
100.0	2.0	1.0	1.179001195	14

$u_n(0)$, obtained by the Bulirsch–Stoer integration technique [1]. Table 5 exhibits more or less typical profiles for this problem. In Table 5 are listed the profiles for $m = 5$, $n = 10$, $b = 2$, $\epsilon = .01$ and $m = 10$, $n = 5$, $b = 2$, $\epsilon = .01$. As a general comment we may state that for the range of m, n, b, and ϵ investigated that the Picard perturbation method solved the problem successfully by producing profiles that matched the analytical solution to a relative error of the order of 10^{-4}, even though $u_0(x)$ is a poor approximation to the true solution. The most obvious exception to this statement is that, when m and n simultaneously took on positive fractional values, the numerical method did not converge or gave absurd results at $x = 0$. This behavior is not surprising, for the following reason. The Lipschitz constant for (3.9) when $m = n = .5$ is $K = \max_{0 \leqslant x \leqslant 1} \max_{Y,y} | .5(1 + \epsilon x^{-.5})/$

TABLE 3

Example 2. $b = 2.0$, $\epsilon = .01$

m	n	$u(0)$ analytical	$u_n(0)$	No. iterations for convergence
0.0	1.0	20.09975124	20.10246977	6
0.5	3.0	14.28285686	nonconvergent	
1.0	0.0	1.01485039	1.01485039	4
1.0	0.5	14.28285686	nonconvergent	
1.0	10.0	14.28285686	14.29446315	8
2.0	3.0	14.28285686	14.28429419	11
2.0	10.0	14.28285686	14.29282009	10
3.0	0.5	14.28285686	nonconvergent	
3.0	2.0	14.28285686	14.28416741	12
5.0	0.5	14.28285686	nonconvergent	
5.0	5.0	14.28285686	14.28600549	7
5.0	10.0	14.28285686	14.29141936	8
10.0	1.0	14.28285686	14.28577433	6
10.0	2.0	14.28285686	14.28497788	12
10.0	5.0	14.28285686	14.28678949	7
10.0	10.0	14.28285686	14.29271511	6

TABLE 4

Example 2. $b = 2.0$, $\epsilon = .0001$

m	n	$u(0)$ analytical	$u_n(0)$	No. iterations for convergence
0.5	0.5	141.4354977	nonconvergent	
0.5	1.0	141.4354977	nonconvergent	
1.0	0.5	141.4354977	nonconvergent	
1.0	1.0	141.4354977	144.4317976	6
1.0	10.0	141.4354977	141.9251559	8
5.0	5.0	141.4354977	141.5821153	7
5.0	10.0	141.4354977	141.7853877	7
10.0	1.0	141.4354977	144.4170685	8
10.0	5.0	141.4354977	141.5866668	7
10.0	10.0	141.4354977	141.7584812	7

TABLE 5

Example 2. $b = 2.0$, $\epsilon = .01$

	$m = 5$, $n = 10$		$m = 10$, $n = 5$	
x	$u(x)$ analytical	$u_8(x)$	$u(x)$ analytical	$u_8(x)$
0.0	14.28285686	14.29141936	14.28285686	14.28678949
0.1	14.28292686	14.29148932	14.28185689	14.28578953
0.2	14.28508689	14.29364801	14.25089342	14.25482605
0.3	14.29926966	14.30782153	14.04196517	14.04589721
0.4	14.34389023	14.35240814	13.29624956	13.30017258
0.5	14.40267260	14.41110174	11.50240365	11.50625752
0.6	14.22495778	14.23320388	8.52355519	8.52704393
0.7	12.84666182	12.85454848	5.37688500	5.37925851
0.8	8.87961901	8.88625330	3.27664815	3.27770938
0.9	4.34777208	4.35027274	2.27407957	2.27439854
1.0	2.00000000	2.00000000	2.00000000	2.00000000

$(x^{.5} + \epsilon Y)(x^{.5} + \epsilon y)|$ and for small x the Lipschitz constant $\approx x^{-.5}/2\epsilon Yy$. For representative values of ϵ, Y, and y (see Tables 2, 3, 4) K is unbounded as $x \to 0$, so that convergence of Picard's method can not be expected.

For this example the convergence criterion was set so that the relative error at $x = 0$ was less than 10^{-4}. With few exceptions the problem was solved iteratively until the convergence criterion was met or until a maximum of 20 iterations, whichever occurred first. For the runs which satisfied the convergence criterion, the entire profile obtained by numerical integration also satisfied the relative error criterion. In general the Picard perturbation method generated after relatively few iterations a profile that was stable and that matched reasonably closely the analytical profile except in the vicinity of the origin. For the runs which did not converge to the analytical result at $x = 0$, but did converge to a constant value, this value had a relative error in the range of .03 to .07.

While Comstock [5] specified that the boundary condition should be $b > 1$, we solved for $b = 1$ as a limiting case. Our numerical experience indicates the need for the condition $b > 1$, for the number of iterations required for $b = 1.0$, $\epsilon = .01$ cases is about twice those for $b = 2.0$, $\epsilon = .01$ for identical m and n. In addition for $b = 1.0$, $\epsilon = .01$ about one-fourth of the cases converged to values of $u_n(0)$ that had relative errors in the .03 to .07 range. On the other hand, for $b = 2.0$, $\epsilon = .01$ or $\epsilon = .0001$ all the cases converged to the analytical result, $u(0)$,

except for those cases with $0 \leqslant m, n \leqslant 1$. In fact convergence was obtained on the average within eight iterations. For all the runs in Tables 2, 3, and 4 we have observed that the Picard iteration method produced the $u_n(0)$ within 1 % of the converged solution in approximately 25 to 50 % fewer iterations than required for convergence. In other words roughly half of the Picard iterations were required to drive $u_n(0)$ over the last 1 % of accuracy.

Since as stated above the Picard perturbation method establishes a fairly stable profile in relatively few iterations except in the neighborhood of the origin, the solution may be speeded up by iterating only over the interval (.10,0). A reduction of step size as much as one-tenth, if needed, would increase accuracy without increasing computer time.

Some comments will be made now about other investigators who have looked at this problem. Comstock [5] stated that with the PLK method, better results were obtained when $m > n$. We have not found any preferred relation $m > n$ or $m < n$ as far as accuracy or number of iterations is concerned for the numerical values investigated.

The problem of C.C. Lin, cited by Comstock [5] which is this example with $m = 0$ and $b = 1$ can not be solved by Lighthill's method according to Lin. The Picard perturbation method however solves Lin's problem with no difficulty and converges to the analytic solution.

EXAMPLE 3 [3, 5]. Carrier has studied equations of the form

$$(x^2 + \epsilon u)(du/dx) + (2/\alpha)\, u = r(x), \tag{3.14}$$

$$u(1) = 1. \tag{3.15}$$

For the case $r(x) = 0$, the equation reduces to a Riccati equation whose analytical solution is given in terms of Bessel functions by

$$x = -(\epsilon u)^{1/2} \frac{\{J_1(\alpha(\epsilon u)^{1/2}) + A Y_1(\alpha(\epsilon u)^{1/2})\}}{\{J_0(\alpha(\epsilon u)^{1/2}) + A Y_0(\alpha(\epsilon u)^{1/2})\}}. \tag{3.16}$$

The constant A is evaluated by solving (3.16) for the boundary condition $u(1) = 1$ at $x = 1$.

The problem for $\alpha = 2$ was solved for several values of ϵ by the Picard perturbation method, using at $\epsilon = 0$, the initial solution

$$u_0 = e^{-(1-1/x)}, \tag{3.17}$$

which incidentally is a poor approximation to the true solution.

Table 6 lists the constant A, the analytical solution $u(0)$, and the Picard pertur-

TABLE 6

Example 3

ϵ	A	$u(0)$	$u_5(0)$
.0010	0.4652609651	2.0265699(10^2)	2.0279958(10^2)
.0005	0.4219184595	3.6329068(10^2)	3.6366690(10^2)
.0001	0.3469146302	1.4554414(10^3)	1.4585247(10^3)

bation solution $u_5(0)$ versus ϵ. The relative error between the analytical and the numerical results is of the order .001.

As a side remark we wish to comment on Carrier's approximation at $x = 0$, which is [3]

$$u(0) \sim [\epsilon \ln(1/\epsilon)]^{-1}. \tag{3.18}$$

For the numerical examples we attacked, we find that Carrier's approximation gives erroneous results compared to both the analytical and the Picard iteration method.

4. Conclusion

We have presented a technique to supplement or complement the conventional approach to perturbation problems. The Picard perturbation method offers an easy to implement and easy to use computational scheme. Numerical experience with the method indicates that the method produces profiles that closely match the analytical solution over the entire interval including the origin. The number of iterations to achieve convergence is quite reasonable. For the examples presented here the convergence properties appear rather remarkable since in each case the initial iterate $u_0(x)$ is a poor approximation to the true profile. Furthermore, the computation may be speeded up dramatically by iterating only in the vicinity of the origin, once the profile has stabilized outside of the neighborhood of the origin. By solving numerically for various ϵ one can generate an interpolation formula for the solution as a function of ϵ, which is equivalent to the expression obtained by conventional perturbation techniques.

Since the analytical solutions are available, the examples in this paper serve as convenient test beds for the PLK and other perturbation methods including ours. This does not mean that the examples could not be solved by nonperturbation techniques. Although other investigators have not mentioned it, these examples may be solved for a wide choice of parameters and boundary conditions by straight

forward integration of the original differential equations, especially if a precision integration method such as Bulirsch–Stoer is employed. There are, however, other cases where the Picard perturbation or other perturbation techniques are required. This may occur, for example, when the choice of the parameters and the boundary conditions are such that the coefficient of the derivative in (1.1) approaches or equals zero. Straight forward integration of the original differential equation may give erroneous results or may exhibit overflow. To illustrate, if we select for Example 1 the parameters $b = -5$, $\eta = 1$, $\epsilon = .10$, the boundary condition is $u(1) = -10$ and the analytical solution at $x = 0$ is $u(0) = 0$. Straightforward integration of the original differential equation with a step size $h = .01$ yields the erroneous results at $x = 0$ of $u(0) = -.1022917$, while the Picard perturbation method for the same step size converges to $u(0) = -.0045833$ after the second iteration.

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REFERENCES

1. R. BULIRSCH AND J. STOER, *Numer. Math.* **8** (1966), 1–13.
2. R. R. BURNSIDE, *SIAM J. Appl. Math.* **18** (1970), 318–321.
3. G. F. CARRIER, *Commun. Pure Appl. Math.* **7** (1954), 11–17.
4. CRAIG COMSTOCK, *SIAM J. Appl. Math.* **16** (1968), 596–602.
5. CRAIG COMSTOCK, *SIAM Rev.* **14** (1972), 433–446.
6. E. L. INCE, "Ordinary Differential Equations," pp. 62–68, reprinted by Dover, New York.
7. M. J. LIGHTHILL, *Quart. J. Mech. Appl. Math.* **1** (1948), 309–318.
8. M. J. LIGHTHILL, *Phil. Mag.* **7** (1949), 1179–1201.
9. W. WASOW, *J. Rational Mech. Anal.* **4** (1955), 751–767.